CS395T: Continuous Algorithms
Homework III

Kevin Tian

Due date: March 13, 2025, start of class (3:30 PM).

Please list all collaborators on the first page of your solutions. Unless we have discussed and I have specified otherwise, homework is not accepted if it is not turned in by hand at the start of class, or turned in electronically on Canvas by then. Send me an email to discuss any exceptions.

1 Problem 1

Let $\mathbf{A} \in \mathbb{R}^{n \times d}$ have rows $\{\mathbf{a}_i\}_{i \in [n]} \in \mathbb{R}^d$, and denote its leverage scores by $\{\boldsymbol{\tau}_i(\mathbf{A})\}_{i \in [n]}$.

(i) Prove that for all $i \in [n]$,
$$\boldsymbol{\tau}_i(\mathbf{A}) = \min_{\substack{\mathbf{x} \in \mathbb{R}^n \\ \mathbf{A}^\top \mathbf{x} = \mathbf{a}_i}} \|\mathbf{x}\|_2^2.$$

(ii) Prove that there is a projection matrix[1] $\mathbf{P} \in \mathbb{S}_{\succeq \mathbf{0}}^{n \times n}$ such that $\boldsymbol{\tau}_i(\mathbf{A}) = \mathbf{P}_{ii}$, for all $i \in [n]$.

2 Problem 2

Let $\mathbf{M} \in \mathbb{S}_{\succ \mathbf{0}}^{d \times d}$ have $\boldsymbol{\lambda}_i(\mathbf{M}) \in [\mu, L]$ for all $i \in [d]$, and let $\epsilon \in (0, 1)$. Prove that there exists a polynomial p of degree $O(\sqrt{\kappa} \log \frac{\kappa}{\epsilon})$, where $\kappa := \frac{L}{\mu}$, such that

$$(1 - \epsilon)\mathbf{M} \preceq p(\mathbf{M})^2 \preceq (1 + \epsilon)\mathbf{M}.$$

You can use without proof that if D is a simply connected region in the complex plane where $0 \notin D$, there is an analytic continuation of $f(z) = \sqrt{z}$ on D that agrees with our definition on $\mathbb{R}_{>0}$.

3 Problem 3

Let $R \geq 0$ and $\mathbf{v} \in \mathbb{R}_{\geq 0}^d$. Give an algorithm which runs in time $O(d \log d)$, and computes[2]

$$\mathrm{argmin}_{\|\mathbf{x}\|_1 \leq R} \left\{ \frac{1}{2} \|\mathbf{x} - \mathbf{v}\|_2^2 \right\}.$$

4 Problem 4

For $\mathbf{K} \in \mathbb{S}_{\succ \mathbf{0}}^{d \times d}$, define the quantity

$$\kappa_{\mathrm{diag}}^\star(\mathbf{K}) := \min_{\substack{\mathbf{W} \in \mathbb{S}_{\succ \mathbf{0}}^{d \times d} \\ \mathbf{W} \text{ is diagonal}}} \frac{\boldsymbol{\lambda}_1\left(\mathbf{W}^{-\frac{1}{2}} \mathbf{K} \mathbf{W}^{-\frac{1}{2}}\right)}{\boldsymbol{\lambda}_d\left(\mathbf{W}^{-\frac{1}{2}} \mathbf{K} \mathbf{W}^{-\frac{1}{2}}\right)}.$$

(i) Prove that $\kappa_{\mathrm{diag}}^\star(\mathbf{K}) \leq \kappa$ iff there is a diagonal matrix $\mathbf{W} \in \mathbb{S}_{\succ \mathbf{0}}^{d \times d}$ such that

$$\mathbf{W} \preceq \mathbf{K} \preceq \kappa \mathbf{W}.$$

---

[1]Recall from Section 2.2, Part VI that $\mathbf{P} \in \mathbb{S}_{\succeq \mathbf{0}}^{n \times n}$ is an orthogonal projection matrix iff $\mathbf{P}^2 = \mathbf{P}$.

[2]Note that this is one step in the projected gradient descent algorithm in Section 3, Part XII. It may be helpful to consider the Karush-Kuhn-Tucker conditions for this constrained optimization problem.

(ii) Prove that if $\mathbf{D} \in \mathbb{S}_{\succ \mathbf{0}}^{d \times d}$ is a diagonal matrix,

$$\kappa_{\mathrm{diag}}^{\star}(\mathbf{K} + \mathbf{D}) \leq \kappa_{\mathrm{diag}}^{\star}(\mathbf{K}).$$

5   Problem 5

Let $f : \mathcal{X} \to \mathbb{R}$ be a convex self-concordant barrier, where $\mathcal{X} \subseteq \mathbb{R}^d$, and assume that f is of Legendre type. Prove that f^* is also self-concordant.[3] You can use without proof that if $\mathbf{M}(t)$ takes $t \in \mathbb{R}$ to invertible matrices,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((\mathbf{M}(t))^{-1}\right) = -(\mathbf{M}(t))^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{M}(t)\right)(\mathbf{M}(t))^{-1}.$$

---

[3]The characterization of convex conjugates immediately following Lemma 3, Part III may be helpful.